

DEPENDENCE SPACES II

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ABSTRACT. This is a continuation of my paper [7] and the presentation on the Conference on Universal Algebra and Lattice Theory, Szeged, Hungary, June 21-25, 2012. The Steinitz exchange lemma is a basic theorem in linear algebra used, for example, to show that any two bases for a finite-dimensional vector space have the same number of elements. The EIS property was introduced by A. Hulanicki, E. Marczewski, E. Mycielski in [4] (see §31 of [3]). In this note we show that its analogue holds in a dependence space.

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1. BASIC NOTION

According to F. Gécseg, H. Jürgensen [2] the result which is usually referred to as the "Exchange Lemma", states that for transitive dependence, every independent set can be extended to form a basis. In [7] discussed some interplay between the discussed notion of [8]–[9] and [1]–[2]. Another proof was presented there, of the result of N.J.S. Hughes [8] on Steinitz' exchange theorem for infinite bases in connection with the notions of transitive dependence, independence and dimension as introduced in [1] and [12]. In the proof we assume Kuratowski-Zorn's Lemma (see [10], [11]), as a requirement pointed in [2]. In this note we extend the results to EIS property known in general algebra as Exchange of Independent Sets Property.

We use a modification of the the notation of [8], [9] and [2]: $a, b, c, \dots, x, y, z, \dots$ (with or without suffices) to denote the elements of \mathbf{S} and $A, B, C, \dots, X, Y, Z, \dots$ for subsets of \mathbf{S} , $\mathbb{X}, \mathbb{Y}, \dots$ denote a family of subsets of \mathbf{S} , n is always a positive integer.

$A \cup B$ denotes the union of sets A and B , $A + B$ denotes the disjoint union of A and B , $A - B$ denotes the difference of A and B , i.e. is the set of those elements of A which are not in B .

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2. DEPENDENT AND INDEPENDENT SETS

The following definition is due to N.J.S. Hughes, invented in 1962 in [8]:

Definition 1. A set \mathbf{S} is called a *dependence space* if there is defined a set Δ , whose members are finite subsets of \mathbf{S} , each containing at least 2 elements, and if the Transitivity Axiom is satisfied.

Definition 2. A set A is called *directly dependent* if $A \in \Delta$.

Definition 3. An element x is called *dependent on* A and is denoted by $x \sim \Sigma A$ if either $x \in A$ or if there exist distinct elements x_0, x_1, \dots, x_n such that

$$(1) \{x_0, x_1, \dots, x_n\} \in \Delta$$

where $x_0 = x$ and $x_1, \dots, x_n \in A$

and *directly dependent on* $\{x\}$ or $\{x_1, \dots, x_n\}$, respectively.

Definition 4. A set A is called *dependent* if (1) is satisfied for some distinct elements $x_0, x_1, \dots, x_n \in A$. Otherwise A is *independent*.

Definition 5. If a set A is *independent* and for any $x \in \mathbf{S}$, $x \sim \Sigma A$, i.e. x is dependent on A , then A is called a *basis of* \mathbf{S} .

Definition 6. TRANSITIVITY AXIOM:

If $x \sim \Sigma A$ and for all $a \in A$, $a \sim \Sigma B$, then $x \sim \Sigma B$.

A similar definition of a *dependence* D was introduced in [1] and [2].

First we recall some definitions of [2], p. 425 and [1]:

Definition 7. Let \mathbf{S} be a set. A *dependence* on \mathbf{S} is a subset D of $2^{\mathbf{S}}$ which has the following property: $X \in D$ if and only if there is a finite non-empty subset X' of X with $X' \in D$. A subset $X \subseteq \mathbf{S}$ is called *dependent* (with respect to D if $X \in D$, otherwise it is called *independent*).

Definition 8. Let \mathbf{S} be a set with dependence D and $a \in \mathbf{S}$, $A \subseteq \mathbf{S}$. The element x is said to *depend on* A if $a \in A$ or there is an independent subset $A' \subseteq A$ such that $A' \cup \{a\}$ is dependent in \mathbf{S} .

Definition 9. The *span* $\langle X \rangle$ of a subset X of \mathbf{S} is the set of all elements of \mathbf{S} which depends on X , i.e. $x \in \langle X \rangle$ iff $x \sim \Sigma X$.

Definition 10. The definitions of [2] and [1] are equivalent to those of [8] in the following sense: every dependence in the sense of [2] and [1] may be expressed as a dependence in the sense of [8].

Proof This follows from the fact that one may start with all finite subsets of a dependent sets of a dependence space in the sense of [2] and [1] to obtain a dependence space in the sense of [8] – with the same dependent (independent) sets.

They define a generalized closure operator $\langle \rangle$ of a finite character (see [6], p. 647.

First we improve the necessity part of the proof of Lemma 8 of [8]:

Lemma 11. *In a dependence space \mathbf{S} the following are equivalent for an element $x \in \mathbf{S}$ and a subset A of \mathbf{S} :*

- (i) $x \sim \Sigma A$ in the sense of [8] and
- (ii) x depends of A in the sense of [2] and [1].

Proof

Assume (i), i.e. that $x \sim \Sigma A$ in the sense of [8].

Therefore, $x \in A$ or there exists elements $x_0 = x$ and $x_1, \dots, x_n \in A$ such that $\{x_0, x_1, \dots, x_n\} \in \Delta$. If the set $\{x_1, \dots, x_n\}$ is dependent in \mathbf{S} in the sense of [8], then let us choose the minimal subset A' of the set $\{x_0, x_1, \dots, x_n\}$ belonging to Δ . Then the set $A' \cup \{x\} = A''$ is independent and (ii) holds. The sufficiency was proved in [7] p. 155.

□

3. STEINITZ' EXCHANGE THEOREM

In Linear Algebra, Steinitz exchange Lemma states that:

if $a \notin \langle A \cup \{b\} \rangle$, then $b \notin \langle A \cup \{a\} \rangle$.

In particular, if A is independent and $a \notin \langle A \rangle$, then:

$A \cup \{a\}$ is independent.

Note, that the following well known properties (see [6]-[8]) are satisfied in a dependence space:

Proposition 12. (2) *Any subset of an independent set A is independent.*

(3) *A basis is a maximal independent set of \mathbf{S} and vice versa.*

(4) *A basis is a minimal subset of \mathbf{S} which spans \mathbf{S} and vice versa.*

(5) *The family (\mathbb{X}, \subseteq) of all independent subsets of \mathbf{S} is partially ordered by the set-theoretical inclusion. Shortly we say that \mathbb{X} is an ordered set (a po-set).*

(6) *Any superset of a dependent set of \mathbf{S} is dependent.*

Lemma 13. *In a dependence space \mathbf{S} , assume that $a \notin \langle A \cup \{b\} \rangle$. Then $b \notin \langle A \cup \{a\} \rangle$.*

Proof

If $b \in \langle A \cup \{a\} \rangle - \langle A \rangle$, then there exists $a_1, \dots, a_n \in A$, such that $b \sim \{a, a_1, \dots, a_n\}$, i.e. $\{a, a_1, \dots, a_n, b\} \in \Delta$.

Therefore $a \in \langle \{b\} \cup A \rangle$, a contradiction. \square

In particular, if A is independent and $a \notin \langle A \rangle$, then:

$A \cup \{a\}$ is independent.

A transfinite version of the Steinitz Exchange Theorem, provides that any independent subset injects into any generating subset.

4. EIS PROPERTY

The EIS (exchange of independent sets) property was introduced by A. Hulanicki, E. Marczewski, E. Mycielski in [4]. First we recall their original definition of EIS property (see [4], [6], p. 647–659). In their paper they use the terminology and notation of [5] (with slight modifications). An *abstract algebra* is a (nonempty) set with a family of fundamental finitary operations. For any nonempty set $E \subset A$. $C(E)$ denotes the subalgebra generated by E , $C(\emptyset)$ is denoting the set of algebraic constants (i.e. the values of the constant algebraic operations). The operation C has finite character, i.e. $C(E) = \bigcup C(F)$, where F runs over the family of all finite subsets of E .

The following theorem about exchange of independent sets is true for all algebras (see [5], p. 58, theorem 2.4 (ii)):

Theorem 14. *Let P, Q and R be subsets of an algebra. If $P \cup Q$ is independent, $P \cap Q = \emptyset$, R is independent, $C(R) = C(Q)$, then $P \cup R$ is independent.*

As the authors of [4] noticed, it might seem at first glance that the relation $C(R) = C(Q)$ could be replaced by a weaker one: $R \subset C(Q)$. Since, as it can be seen from the results of [4], this is not generally true, the authors say that an algebra satisfies *the condition of exchange of independent sets* (EIS) whenever for any subsets P, Q and R of it, the relations: $P \cup Q$ is independent, $P \cap Q = \emptyset$, R is independent and $R \subset C(Q)$ imply that $P \cup R$ is independent.

We transform the original definition of EIS property from *algebras* to *dependence spaces* in the natural way:

Definition 15. A dependence space \mathbf{S} satisfies the EIS property, if for arbitrary subsets P, Q and R of \mathbf{S} the conditions:

- (7) $P \cap Q = \emptyset$;
 - (8) $P \cup Q$ is an independent set in \mathbf{S} ;
 - (9) R is an independent set in \mathbf{S} , $R \subseteq \langle Q \rangle$;
- altogether imply that:
- (10) $P \cup R$ is an independent set.

Theorem 16. *In a dependence space \mathbf{S} , the EIS property holds.*

Proof

Assume (7) – (9).

To show (10) assume a contrario that $P \cup R$ is a dependent set. Therefore there exist (all different) elements $a_1, \dots, a_n, b_1, \dots, b_m \in P \cup R$ with $a_1, \dots, a_n \in P$ and $b_1, \dots, b_m \in R$ and such that $\{a_1, \dots, a_n, b_1, \dots, b_m\} \in \Delta$. From (7) and (9) it follows that there exists an element $a_1 \in P$ such that $a_1 \sim \Sigma\{a_2, \dots, a_n, b_1, \dots, b_m\}$, i.e. $a_1 \sim \Sigma((P - \{a_1\}) \cup R)$. But for very element $b \in R$, $b \sim \Sigma Q$, therefore $b \sim \Sigma((P - \{a_1\}) \cup Q)$. Moreover, $c \sim \Sigma((P \cup Q) - \{a_1\})$, for every $c \in ((P - \{a_1\}) \cup R)$. Thus, by the transitivity axiom $a_1 \sim \Sigma((P \cup Q) - \{a_1\})$. That contradicts (8), as $a_1 \in P \cup Q$ and it is clear, that for an independent set A , one gets for each $a \in A$, that $a \notin \langle A - \{a\} \rangle$. \square

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